THE KNOX SCHOOL

# AP Calculus BC
# 2020 Summer Assignment

**Directions:** You must show all work, even for multiple choice. Any graphing problem should be done without a graphing calculator.

**Due Date**: First day of school! You will be held accountable for this material upon your return to school. Yes, that means a test or a quiz on this material is going to happen.

The next page and a half contain properties and rules that you should know off the top of your head. I included them for your reference in case you need a refresher.

## Properties of Exponents and Logarithms

### Exponents

Let $a$ and $b$ be real numbers and $m$ and $n$ be integers. Then the following properties of exponents hold, provided that all of the expressions appearing in a particular equation are defined.

1. $a^m a^n = a^{m+n}$ 2. $(a^m)^n = a^{mn}$ 3. $(ab)^m = a^m b^m$

4. $\dfrac{a^m}{a^n} = a^{m-n},\ a \neq 0$ 5. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m},\ b \neq 0$ 6. $a^{-m} = \dfrac{1}{a^m},\ a \neq 0$

7. $a^{\frac{1}{n}} = \sqrt[n]{a}$ 8. $a^0 = 1,\ a \neq 0$ 9. $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

where $m$ and $n$ are integers in properties 7 and 9.

**Properties of Logarithms** (Recall that logs are only defined for positive values of $x$.)

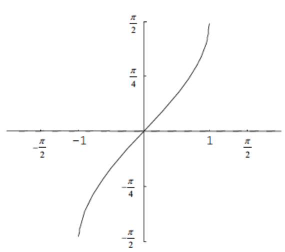
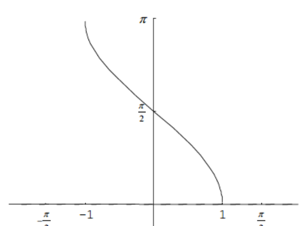
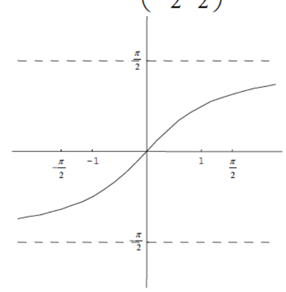
| For the natural logarithm | For logarithms base $a$ |
|---|---|
| 1. $\ln xy = \ln x + \ln y$ | 1. $\log_a xy = \log_a x + \log_a y$ |
| 2. $\ln \frac{x}{y} = \ln x - \ln y$ | 2. $\log_a \frac{x}{y} = \log_a x - \log_a y$ |
| 3. $\ln x^y = y \cdot \ln x$ | 3. $\log_a x^y = y \cdot \log_a x$ |
| 4. $\ln e^x = x$ | 4. $\log_a a^x = x$ |
| 5. $e^{\ln x} = x$ | 5. $a^{\log_a x} = x$ |

**Useful Identities for Logarithms**

| For the natural logarithm | For logarithms base $a$ |
|---|---|
| 1. $\ln e = 1$ | 1. $\log_a a = 1$, for all $a > 0$ |
| 2. $\ln 1 = 0$ | 2. $\log_a 1 = 0$, for all $a > 0$ |

Please be familiar with these graphs. More so, the domain and range of the graphs.

**GRAPHS OF INVERSE TRIG FUNCTIONS**

| | | |
|---|---|---|
| Domain: $[-1,1]$ | Domain: $[-1,1]$ | Domain: $(-\infty,\infty)$ |
| Range: $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ | Range: $[0,\pi]$ | Range: $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ |
|  |  |  |
| $f(x)=\sin^{-1} x$ | $f(x)=\cos^{-1} x$ | $f(x)=\tan^{-1} x$ |
| $f(x)=\arcsin x$ | $f(x)=\arccos x$ | $f(x)=\arctan x$ |

In addition to the above material, you also know the 3 Pythagorean identities, the reciprocal trig identities, and all trig values of special angles (unit circle).

Find the integral of the following functions. The answers for #1-4 involve inverse sine or inverse tangent.

1. $\int_0^1 \frac{4}{\sqrt{16-x^2}} dx =$

2. $\int \frac{e^x}{\sqrt{1-e^{2x}}} dx =$

3. $\int_0^2 \frac{1}{2x^2+8} dx$

4. $\int \frac{x}{x^4+16}\,dx =$

5. $\int_1^4 \frac{4x^4-5x^2-8x+1}{2\sqrt{x}}\,dx$

6. $\int \frac{\sin^3(2\,x)\cos\,(2x)}{\sqrt{4+\sin^4(2\,x)}}\,dx$

7. $\int \frac{\ln(\ln(x))}{x ln(x)} dx$

8. $\int_0^3 (2x-3)^3 dx$

Find the derivative of the following:

9. $f(x) = 3\sin^{-1}(2x), \quad f'(x) =$

10. $y = \frac{1}{2}\tan^{-1}(\frac{\theta}{3})$, $y'(3) =$

11. $f(x) = -2x^3\cos^{-1}(5x)$, $f'(x) =$

12. $y = -e^{2x}\sin^{-1}\left(\frac{1}{2}x - 1\right)$, $y' =$

13. $y = (3x-1)^3(4x^2-3)^2$. Express answer in simplest factored form.

14. $g(x) = \frac{(8x+3)^2}{(7x-2)^{\frac{3}{2}}}$ Express answer in simplest factored form.

15. $y = \ln(4x)\tan(3x^2)$

16. $2xy^2 - 3x = 5xy$. Find $\frac{dy}{dx}$ (Hint: implicit)

Limits and Continuity.

17. $lim_{x \to 1^+} \frac{|5-5x|}{x-1} =$

18. $lim_{x \to 0} \frac{sinx-cosx}{cos2x} =$

19. $lim_{x \to -\infty} \frac{4x^3-6x+1}{7x^4+5} =$

20. $lim_{x \to 0} \frac{\sin(cosx)}{secx}$

21. $lim_{x \to 0^-} \frac{3}{1+e^{\frac{1}{x}}} =$

22. For the following function, find any discontinuities. State whether they are removable or non-removable. If removable, redefine the function so that it will be continuous at that value.

$$f(x) = \begin{cases} 3x - 1; & x > 2 \\ -5; & x = 2 \\ 1 + 2x; & x < 2 \end{cases}$$

23. $lim_{x \to -\infty} \frac{3-x}{\sqrt{2x^2+1} + 5x} =$

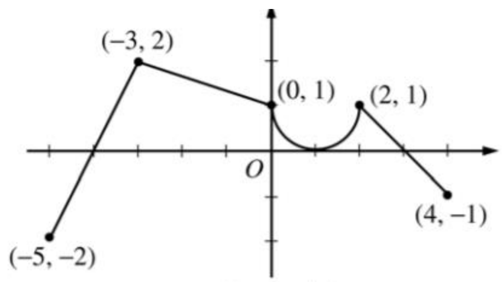
24. Find the values of $a$ and $b$ that will make the function $f(x)$ differentiable.

$$f(x) = \begin{cases} ax^2 + 1 & x \geq 1 \\ bx - 3 & x < 1 \end{cases}.$$

25. Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.

26. Find equations of both lines that are tangent to the curve $y = 1 + x^3$ and are parallel to the line $12x - y = 1$.

27.



Graph of $f$

The graph of the function $f$ shown above consists of a semicircle and three line segments. Let $g$ be the function given by $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(0)$ and $g'(0)$.

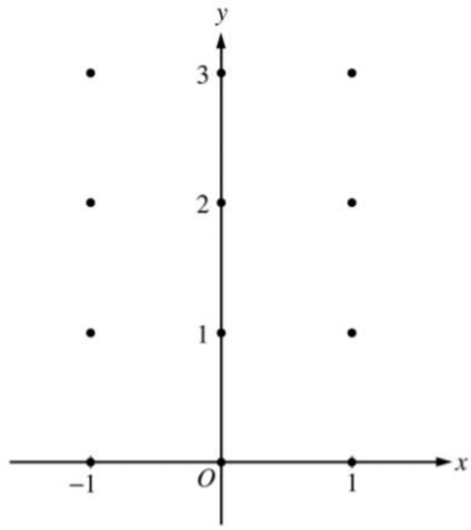
(b) Find all values of $x$ in the open interval $(-5, 4)$ at which $g$ attains a relative maximum. Justify your answer.

(c) Find the absolute minimum value of $g$ on the closed interval $[-5, 4]$. Justify your answer.

(d) Find all values of $x$ in the open interval $(-5, 4)$ at which the graph of $g$ has a point of inflection.

**28.** Consider the differential equation $\frac{dy}{dx} = x^2(y-1)$.

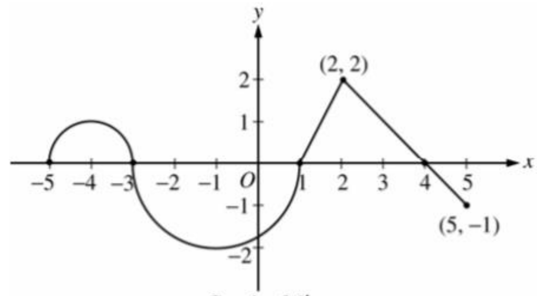
(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.
**(Note: Use the axes provided in the pink test booklet.)**




(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the $xy$-plane. Describe all points in the $xy$-plane for which the slopes are positive.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = 3$.

29. Let $f$ be a function defined on the closed interval $-5 \le x \le 5$ with $f(1) = 3$. The graph of $f'$, the derivative of $f$, consists of two semicircles and two line segments, as shown above.



Graph of $f'$

(a) For $-5 < x < 5$, find all values $x$ at which $f$ has a relative maximum. Justify your answer.

(b) For $-5 < x < 5$, find all values $x$ at which the graph of $f$ has a point of inflection. Justify your answer.

(c) Find all intervals on which the graph of $f$ is concave up and also has positive slope. Explain your reasoning.

(d) Find the absolute minimum value of $f(x)$ over the closed interval $-5 \le x \le 5$. Explain your reasoning.

30. Consider the curve given by $x^2 + 4y^2 = 7 + 3xy$.

(a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

(b) Show that there is a point $P$ with $x$-coordinate 3 at which the line tangent to the curve at $P$ is horizontal. Find the $y$-coordinate of $P$.

(c) Find the value of $\frac{d^2y}{dx^2}$ at the point $P$ found in part (b). Does the curve have a local maximum, a local minimum, or neither at the point $P$? Justify your answer.

31. A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time $t$ minutes, where $v$ is a differentiable function of $t$. Selected values of $v(t)$ for $0 \le t \le 40$ are shown in the table above.

| $t$ (min) | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 |
|---|---|---|---|---|---|---|---|---|---|
| $v(t)$ (mpm) | 7.0 | 9.2 | 9.5 | 7.0 | 4.5 | 2.4 | 2.4 | 4.3 | 7.3 |

(a) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to approximate $\int_0^{40} v(t)\,dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t)\,dt$ in terms of the plane's flight.

(b) Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.

(c) The function $f$, defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \le t \le 40$. According to this model, what is the acceleration of the plane at $t = 23$ ? Indicates units of measure.

(d) According to the model $f$, given in part (c), what is the average velocity of the plane, in miles per minute, over the time interval $0 \le t \le 40$?

# Important Theorems in Differential Calculus

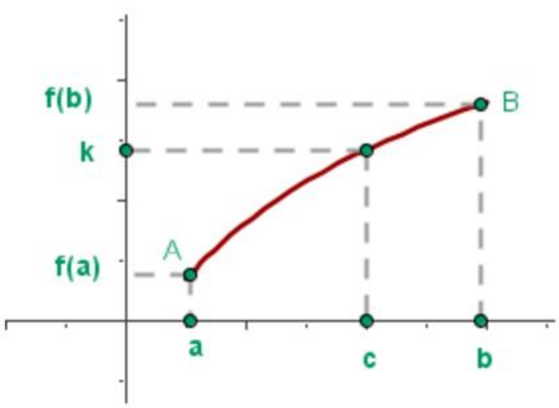
The following theorems are important ones in regards to differential calculus. However, the most important one from the four below is the Mean Value Theorem.

Make sure that you verify the conditions (hypotheses) of the following theorems before attempting to solve for anything.

**<u>Intermediate Value Theorem</u> (IVT)**-If f is continuous on a closed interval [a,b] and **k** is any number between f(a) and f(b), inclusive, then there is at least one number **c** in the interval [a,b] such that f(x)=k.

In normal language it basically means if I have a continuous function on [a,b] and I know the y value at x=a, and I know the y value at x=b, then f must hit every y value in between (and including) those y values. The whole theorem is based on continuity.

**See picture below.**



**Ex.** Show that the function $f(x) = x^2 - 2x - 3$ has a zero somewhere on the interval [-2, 1].

First, verify that $f(x)$ is continuous on the given interval. It is continuous because all polynomials are continuous everywhere.

Next, find $f(-2)\ and\ f(1)$.

$f(-2) = 5 \quad and\ f(1) = -4$. Therefore, at some x value, c, $f(c)$ must equal 0.

Now, we set up an equation saying that.

$$f(c) = 0$$
$$c^2 - 2c - 3 = 0.$$

It is not mandatory to use c, it is just an x value so you can just keep x in the problem if you'd prefer.

Now, solve for c. $(c-3)(c+1) = 0, so\ c = -1\ and\ c = 3$. Throw out $c = 3$ because it is outside the given interval. Therefore $f(-1) = 0\ and\ c = -1$.

**Extreme Value Theorem (EVT)-**If a function f is *continuous* on a finite *closed interval* [a,b], then f has both an absolute maximum and an absolute minimum.

This should be a fairly obvious theorem. Think of a rollercoaster and use the length of the ride as the closed interval [a,b]. At some point, either at the endpoints or within the time [a,b], the rollercoaster must reach its highest and its lowest point. This is all the theorem says. I italicized the words continuous and closed interval because if either of those two conditions fail, the theorem is not guaranteed to hold true.

You did many examples proving this theorem when you found absolute maximums and absolute minimums on a closed interval (curve sketching).

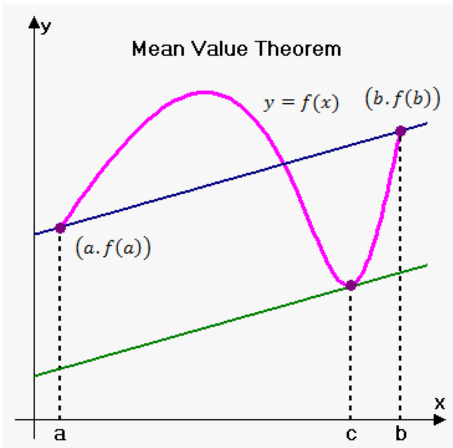
**Mean Value Theorem (MVT)-** Suppose $f(x)$ is a function that satisfies both of the following.

1. $f(x)$ is continuous on the closed interval [*a*,*b*].
2. $f(x)$ is differentiable on the open interval (*a*,*b*).

Then there is a number *c* such that *a* < *c* < *b* and

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

In normal language, this means that if $f(x)$ is $both\ continuous$ on the closed interval $and\ differentiable$ on the open interval, then, at some value in between *a* and *b*, the secant line is parallel to the tangent line….making their slopes equal (set the derivative equal to the normal slope you learned when you were a little kid). See picture below.




Ex. Verify that the hypotheses (conditions) of the MVT are satisfied on the given interval, and find all values c in that interval that satisfy the conclusion of the theorem.

$$f(x) = x^3 + x - 4, \ [-1, 2]$$

First, verify the hypotheses hold true:

1. $f(x)$ is continuous on [-1, 2] since $f(x)$is a polynomial and continuous everywhere.
2. $f'(x) = 3x^2 + 1$ which has no values where it is undefined. Therefore, $f'$ is differentiable on the interval.

Conclusion of the theorem:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

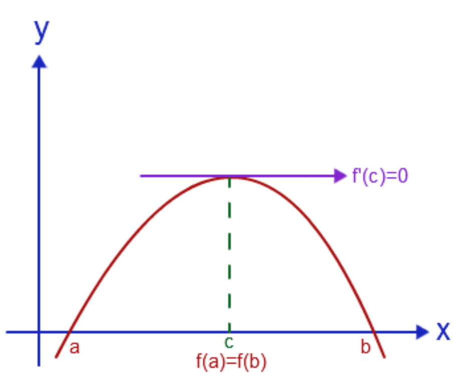
$$3c^2 + 1 = \frac{6-(-6)}{2-(-1)}$$

$$3c^2 + 1 = 4$$

$$c = \pm 1,$$

Use only $c = +1$ since $c$ must be in the open interval (*a, b*).

**Rolle's Theorem-**This theorem is just a special case of the MVT. It has this additional condition: $f(a) = f(b)$, therefore, at some c in (a,b), $f'(c) = 0$. See picture below. In the picture, both $f(a)\ and\ f(b)$ equal 0…they don't have to both equal 0, just both be equal to each other.




In normal language it means that if the secant line connection x=a and x=b has a slope of zero (since $f(a)\ and\ f(b)$ are equal), then at some value c, in the open interval (a,b), the derivative will also equal 0.

Ex. For the function below, verify that the hypotheses (conditions) of Rolle's Theorem (the same as MVT plus the additional one) are satisfied on the given interval, and find all values c in that interval that satisfy the conclusion of the theorem ( $f'(c) = 0$ ).

$$f(x) = x^2 - 8x + 15,\ \ [1,7].$$

Hypotheses satisfied:

1. $f(x)$ is continuous on [1, 6] since it is a polyomial and polynomials are continuous everywhere.
2. $f(x)$ is differentiable on (a,b)
   $f'(x) = 2x - 8$ Since there are no values where $f'(x)$ is undefined, it is differentiable on the given interval.
3. $f(1) = 8 \; and \; f(7) = 8$

Conclusion:

$$f'(c) = 0$$
$$2c - 8 = 0$$
$$c = 4$$

(Confirm that c is in the interval (3,5), which it is.)

Practice problems.

1. Decide whether or not the Intermediate Value Theorem holds for the following equation on the given interval. If it does, find the x value where the function equals 0.

$$f(x) = \frac{3x}{x-4}; \quad [0, 5]$$

2. Use the IVT to prove that $x^2 + 3x - 2 = 0$ has one real solution between 0 and 1.

3. Verify that the hypotheses of the MVT are satisfied on the given interval, and final all values c that satisfy the conlusion of the theorem.

a. $f(x) = x^3 + x - 4;\ \ [-1, 2]$

b. $f(x) = x - \frac{1}{x};\ \ [3, 4]$

4. Verify that the hypotheses for Rolle's Theorem are satisfied on the given interval, and find all values c that satisfy the conclusion of the theorem.

a. $f(x) = cosx;\ \ \left[\frac{\pi}{2}, \frac{3\pi}{2}\right].$

b. $f(x) = \frac{1}{2}x - \sqrt{x}$; $[0, 4]$